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Two-Level Open Quantum Systems: Energy Dissipation, Population Dynamics, and Entropy Evolution

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1 ABSTRACT

We examine the dynamics of a two-level system (TLS) in an open quantum system under three decoherence models: amplitude damping, thermalization, and depolarizing channel. Specifically, we analyze their impact on the system's energy, population, and entropy evolution. Our results indicate that amplitude damping leads to exponential energy decay as the system relaxes to the ground state, while the thermalization process drives the system to a steady-state energy determined by the bath temperature. The depolarizing channel induces rapid state mixing, causing energy to decay to zero. Population dynamics reveal that all three models drive the system toward distinct steady states, with depolarization leading to a maximally mixed state. Entropy evolution varies as it stabilizes for amplitude damping and thermalization but reaches its maximum in the depolarizing channel. These findings provide insights into energy dissipation, population redistribution, and entropy changes in TLS under different system-environment interactions.

2 INTRODUCTION

A two-level system (TLS) is one of the fundamental models in quantum mechanics which serves as a cornerstone for understanding a wide array of quantum phenomena [1]. TLS consists of two distinct states. These states can represent various physical scenarios, such as the ground and excited states of an atom, spin-up and spin-down states of an electron, or two distinct energy levels of a quantum dot [2, 3]. TLS plays a pivotal role as the qubit in quantum computing - the basic unit of quantum information. Moreover, TLS models are also employed in quantum optics for the treatment of light-matter interactions [4]. Additionally, TLSs are used to understand processes in quantum thermodynamics [5] and quantum biology [6].

Given the importance of TLSs in various quantum applications, it is crucial to consider them within the framework of open quantum systems, where the dynamics of a quantum-mechanical system that interacts with the environment are studied [7]. In general, these interactions significantly change the dynamics of the system and result in quantum dissipation. This very action causes the information contained in the system to be lost to its environment [1].

In this study, we attempt to investigate the impact of environmental influence on a

TLS. For this, we apply three well known decoherence models—amplitude damping [2], thermalization process [8], and depolarizing channel [9]—on a typical TLS and analyze resulting dynamics. Moreover, we utilize the Lindblad master equation to determine the time evolution of the density operator of the TLS under the influence of the aforementioned decoherence models. The paper is organized in the following way: In the first section, a detailed mathematical description of a TLS is provided. The second section is dedicated to studying the dynamics of TLS under decoherence. The results of our analysis are presented in the third section, followed by conclusions drawn from these findings.

3 MATHEMATICAL DESCRIPTION OF A TLS

The mathematical description of a TLS is expressed using the framework of a two dimensional Hilbert space $\mathcal{H}$. The general state of a TLS $|\psi_t\rangle$ at time t can be expressed as a linear combination (or superposition) of the basis states $|0\rangle$ and $|1\rangle$ of the Hilbert space $\mathcal{H}$ given by $|\psi_t\rangle = \alpha_t|0\rangle + \beta_t|1\rangle$where $t \in \mathbb{N}$ is the time parameter, α_t and β_t are complex coefficients satisfying the normalization condition $|\alpha_t|^2 + |\beta_t|^2 = 1$. In the Bloch sphere representation [2], the state of a TLS can be represented in an alternative way as $|\psi_t\rangle = cos\theta_t|0\rangle + e^{i\phi_t}sin\theta_t|1\rangle$ where $t \in \mathbb{N}$ is the time parameter $\theta_t \in [0,\frac{\pi}{2}]$ and $\phi_t \in [0,2\pi)$. The dynamics of a TLS, in a closed environment, are governed by the Schrödinger equation, $i\hbar\frac{d}{dx}|\psi_t\rangle = H|\psi_t\rangle$ where H is the Hamiltonian of the system and $\hbar$ is the reduced Plank constant. The most common form of the free Hamiltonian H_0 for a two-level system, which is free from environmental influence, is given by $H_0 = -\frac{\hbar\omega}{2}\sigma_z$where ω is the transition frequency between the two energy levels and $\sigma_z$is the Pauli-Z operator [10]. Accordingly, the states $|0\rangle$ and $|1\rangle$ can be interpreted as the ground and excited states of the TLS with the energies of $-\frac{\hbar\omega}{2}$and $\frac{\hbar\omega}{2}$ respectively.

Two-level systems that interact with the environment give rise to statistical mixtures of states [11]. Such systems are called open TLSs. The usual ket and bra vector formalism cannot be used to represent such mixtures of states. Hence, in general, a mathematical entity called density matrix is used to represent the states of an open quantum system [12]. Therefore, in our study, we utilize the density matrix formalism to represent the states of the open TLSs. Moreover, open quantum systems often deal with non-unitary evolution that cannot be described by the Schrödinger equation alone. Hence, to describe the dynamics of open quantum systems, more sophisticated formalism is required. In many cases, the dynamics of an open quantum system are described using the Markov approximation, which assumes that the environment immediately returns to equilibrium after interactions with the quantum system [13]. The Lindblad equation, the most well-known master equation derived from the Markov approximation, is a key component in describing the dynamics of open quantum systems. [14, 15]. The time evolution of the density matrix ρ_t at time t in the presence of environmental interactions, can be described by the Lindblad master equation given by

$$\frac{d\rho_t}{dt} = -\frac{i}{\hbar}[H_0, \rho_t] + \sum_k L_k\rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_t\} \tag{1}$$

where L_k are the Lindblad operators characterize the decoherence processes, H_0 is the free Hamiltonian of the TLS and $\{a, b\} = ab + ba$ is the anti-commutator. Operators L_k are sometimes called jump operators. These operators are critical for modeling various physical phenomena in quantum optics and quantum thermodynamics, enabling the accurate simulation of systems interacting with their environments [13].

4 TLS UNDER DECOHERENCE

In our research, we focus on the dynamics of a TLS under 3 different decoherence models: amplitude damping, thermalization process and depolarizing channel. Accordingly, we choose appropriate Lindblad operators to mimic the dynamics under the decoherence models we consider. By analyzing these models, we aim to examine the mechanisms of decoherence and their implications for the stability and performance of quantum systems. Amplitude damping models the loss of energy from the system to the environment, such as spontaneous emission [2]. The corresponding Lindblad operator is given by $L_1^{(a)} = \sqrt{\gamma}\sigma_-$ where $\sigma_- = |0\rangle\langle 1|$ is the lowering operator and γ is the rate of energy dissipation. The thermalization process models the thermal relaxation where the system can exchange energy with a thermal bath at non-zero temperature [8]. The corresponding Lindblad operators are given by $L_1^{(th)} = \sqrt{\mu(1 + n_{th})}\sigma_-$ and $L_2^{(th)} = \sqrt{\mu n_{th}}\sigma_+$ where $\sigma_+ = \sigma_-^\dagger$is the raising operator, n_{th} is the average number of thermal excitation in the bath and μ is the rate of energy dissipation. Depolarizing channel models the process where the state of the TLS is replaced by a completely mixed state with a certain probability [9]. The corresponding operators are given by $L_1^{(d)} = \sqrt{\frac{\lambda}{3}}\sigma_x$, $L_2^{(d)} = \sqrt{\frac{\lambda}{3}}\sigma_y$ and $L_3^{(d)} = \sqrt{\frac{\lambda}{3}}\sigma_z$ where σ_x, σ_y and σ_z are Pauli operators and λ is the depolarizing rate. We substitute Lindblad operators corresponding to each of the decoherence model to the Lindblad master equation in (1) and solve the matrix differential equation to determine the density operator of the TLS at time t. For the initial state of the TLS, we employ the pure state of $\rho_0 = |\psi_0\rangle\langle\psi_0|$ where $|\psi_0\rangle = cos\theta_0|0\rangle + sin\theta_0|1\rangle$ for some $\theta_0 \in [0, \frac{\pi}{2}]$. Further, we calculate the expected energy $\langle H\rangle$ of TLS as a function of time t under the aforementioned three decoherence models by using the formular of $\langle H\rangle = Tr(\rho_t H_0)$ where $Tr(\,.\,)$ is the trace of a matrix, H_0 is the free Hamiltonian of TLS and ρ_t is the density operator of the TLS at time t. We obtained the following expressions for the $\langle H\rangle$ under different decoherence models as

$$\langle H\rangle_f = -\frac{\hbar\omega}{2}\cos 2\theta_0$$
$$\langle H\rangle_a = -\frac{\hbar\omega}{2}(1 - 2e^{-\gamma t}\sin^2\theta_0)$$
$$\langle H\rangle_{th} = -\frac{\hbar\omega}{2}\left(\frac{1 + 2e^{-(1+2n_{th})t\mu}[n_{th}\cos^2\theta_0 - (n_{th} + 1)\sin^2\theta_0]}{1 + 2n_{th}}\right) \qquad (2)$$
$$\langle H\rangle_d = -\frac{\hbar\omega}{2}e^{-\frac{2\lambda}{3}t}\cos 2\theta_0$$

the expected values of energy of the TLS under amplitude damping, thermalization process and depolarization channel respectively. Figure 1 shows the variation of the expected value of the energy of the TLS as time progresses under different decoherence models for a particular case.

5 POPULATION AND ENTROPY EVOLUTION

In this section, we analyze the population evolution of the TLS under the three decoherence models. Moreover, we investigate the entropy dynamics of TLS to qualitatively describe

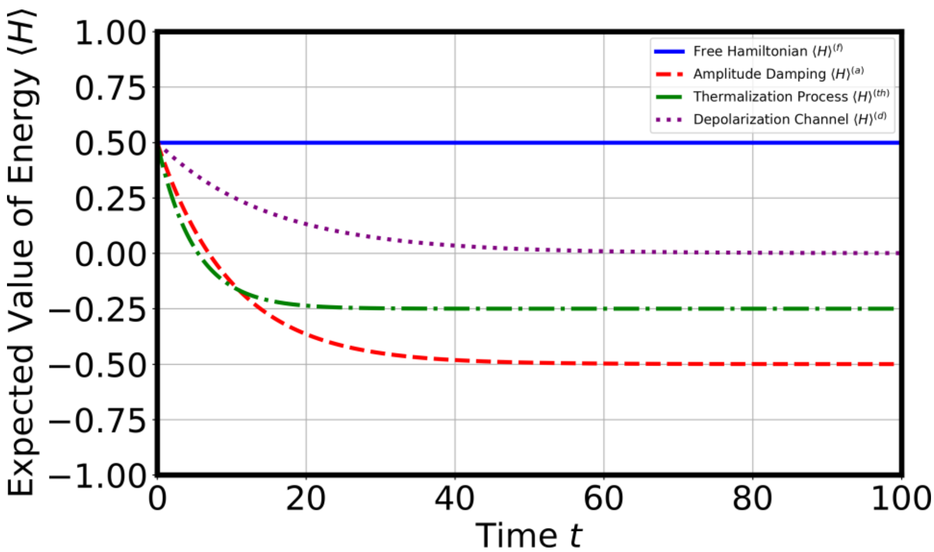

Figure 1: Time evolution of the expected value of energy $\langle H \rangle$ of TLS for $t = 100$ under free Hamiltionion (blue solid line), amplitude damping (red dashed line), thermalization process (green dash-dotted line) and depolarization (purple dotted line) where the TLS is initiated from the excited state (i.e. $|\psi_0\rangle = |1\rangle$). For simplicity, reduced Plank's constant $\hbar$ and the transition frequency ω were set to 1. The decoherence parameters of γ, μ and λ were set to 0.1. The average number of thermal excitation n_{th} in the bath was taken as 0.5.

the disorder introduced by environmental interactions. These analyses provide a deeper insight into the effects of decoherence mechanisms and the energy dissipation of TLS. The population of the TLS states, given by the diagonal elements of the density matrix ρ_t, represents the probability of the system being in either the ground state $|0\rangle$ or the excited state $|1\rangle$. The populations are defined as $P_0(t) = \rho_{00}(t)$ and $P_1(t) = \rho_{11}(t)$ where $P_0(t)$ and $P_1(t)$ denote the probabilities of the TLS being in the ground and excited states respectively and $\rho_{00}(t)$ and $\rho_{11}(t)$ are the diagonal elements of the density matrix at time t. The population dynamics under the three decoherence models are obtained by solving the Lindblad master equation. Like in the previous section, we employ the pure state of $\rho_0 = |\psi_0\rangle\langle\psi_0|$ where $|\psi_0\rangle = cos\theta_0|0\rangle + sin\theta_0|1\rangle$ for some $\theta_0 \in [0, \frac{\pi}{2}]$. The population evolution under different decoherence models is summarized in Table 1.

Entropy quantifies the loss of coherence and purity of the quantum state due to environmental interactions. For a quantum-mechanical system described by a density matrix ρ the von Neumann entropy is given by $S = -Tr(\rho\, ln\rho)$ where $Tr(.)$ denotes the trace and $\ln(.)$ denotes the matrix version of the natural logarithm [16]. In our study, we have numerically calculated the entropy of the TLS under the three different decoherence models for time values $t = 2n$ where $n \in \{1, .., 50\}$. In this calculation, the initial state of the system was taken as $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The parameters γ, μ and λ were set to 0.1. The average number of thermal excitation n_{th} in the bath was taken as 0.5.

6 DISCUSSION

Our analysis reveals distinct energy dissipation behaviors for each decoherence model. From the expression of $\langle H\rangle_f$ in (2), the average energy of the free TLS attains a minimum

Table 1: Population evolution of TLS under three different decoherence models. The initial state of the TLS is taken as $|\psi_0\rangle = cos\theta_0|0\rangle + sin\theta_0|1\rangle$ for some $\theta_0 \in [0, \frac{\pi}{2}]$.

Decoherence Model	Population Evolution
Amplitude Damping	$P_0(t) = 1 - e^{-\gamma t}\sin^2\theta_0$ $P_1(t) = e^{-\gamma t}\sin^2\theta_0$
Thermalization Process	$P_0(t) = \frac{1+n_{th}}{1+2n_{th}} + \frac{n_{th}\cos^2\theta_0 - (1+n_{th})\sin^2\theta_0}{1+2n_{th}} e^{-(1+2n_{th})\mu t}$ $P_1(t) = \frac{1+n_{th}}{1+2n_{th}} - \frac{n_{th}\cos^2\theta_0 - (1+n_{th})\sin^2\theta_0}{1+2n_{th}} e^{-(1+2n_{th})\mu t}$
Depolarizing Channel	$P_0(t) = \frac{1}{2}\left(1 + e^{-\frac{2\gamma}{3}t}\cos 2\theta_0\right)$ $P_1(t) = \frac{1}{2}\left(1 - e^{-\frac{2\gamma}{3}t}\cos 2\theta_0\right)$

and a maximum of $-\frac{\hbar\omega}{2}$ and $\frac{\hbar\omega}{2}$ for $\theta_0 = 0$ and $\theta_0 = \frac{\pi}{2}$ respectively. In other words, the minimum and maximum average energy values are obtained when the TLS is initialized with the states of $|\psi_0\rangle = |0\rangle$ and $|\psi_0\rangle = |1\rangle$ respectively. This is compatible with the ground and excited states of the TLS. That is, when the TLS is initialized in ground state it remains in the ground state under the evolution of free Hamiltonian. The same holds true for the excited state, as shown in Figure 1. Moreover, at t = 0, the expected value of the energy of the TLS under all the decoherence models is equal to that of the free TLS. The expression of $\langle H\rangle_a$ in (2) describes the average energy under the amplitude damping model. The term $e^{-\gamma t}$ represents the exponential decay of energy over time t. The factor $\sin^2\theta_0$ indicates the dependence on the initial state. As t increases, $e^{-\gamma t}$ approaches zero, causing the average energy to approach $-\frac{\hbar\omega}{2}$ which is the energy of the ground state. The average energy, when the TLS interacts with a thermal bath, is given by the expression of

$\langle H \rangle_{th}$ in (2). The exponential term $e^{-(1+2n_{th})\mu t}$ represents the thermalization process over time. The numerator captures the interaction of the system's initial state with the thermal environment. As t goes to infinity, the system reaches thermal equilibrium, and the average energy stabilizes according to the temperature of the bath. In depolarizing channel, the term $e^{-\frac{2\gamma}{3}t}$ governs the impact of depolarization on the average energy of TLS as shown in the expression of $\langle H \rangle_d$ in (2). In the asymptotic limit (i.e. when $t \rightarrow \infty$), the average energy of the TLS, under depolarization, decays towards zero. This result makes sense because, over time, the depolarizing channel tends to mix the states, resulting in a completely mixed state for the density matrix of the TLS [17]. As a result, the system is equally likely to be in either of the two energy levels. The positive and negative contributions to energy cancel each other out, yielding an average energy of zero, as evident in Figure 1.

The effect of decoherence parameters and system-related properties on energy dissipation can be identified from the expressions given in (2). The energy loss rate in amplitude damping is dictated by the decay rate γ, which determines how quickly the system relaxes to the ground state. In the thermalization process, both the thermalization rate μ and the bath temperature (characterized by n_{th}) influence the steady-state energy; a higher n_{th} results in the system retaining more energy at equilibrium. The depolarizing channel, governed by the depolarization rate λ, leads to a uniform mixing of states, causing the energy to decay to zero regardless of the initial state. Additionally, the initial state of the system significantly impacts energy loss, particularly in amplitude damping and thermalization, where states closer to the excited state experience faster dissipation. These observations emphasize that decoherence-induced energy loss is not only dependent on the environmental parameters but also on intrinsic system properties, such as the initial quantum state and the nature of system-environment interactions.

The explicit expressions of the population of TLS under the three decoherence models are tabulated in Table 1. According to Table 1, for long times, all three decoherence models exhibit a tendency for the TLS to relax toward a steady state. In the amplitude damping model, the system evolves such that one population decays exponentially while the other increases correspondingly in the long-time limit. Specifically, $P_1(t)$ decreases over time, while $P_0(t)$ increases, reflecting the transfer of population from the excited state to the ground states due to energy dissipation. This behavior is governed by the decay rate γ which determines how quickly the system relaxes to its steady-state populations. In the thermalization process, both populations approach their respective equilibrium values depending on the temperature of the environment, reaching a steady state governed by the thermal bath. Under the depolarizing channel, as time progresses, the system approaches a maximally mixed state, where the populations of the two states become equal. This indicates the loss of any initial coherence. The depolarizing channel behaves similarly to a statistical mixture, where the system's dynamics are governed by a random, probabilistic mixing of states, rather than by quantum coherence.

The entropy evolution exhibits distinct behaviors under different decoherence mechanisms. In the amplitude damping model, entropy initially increases as the system loses coherence but eventually stabilizes when the system reaches the ground state. The thermalization process, influenced by the surrounding bath temperature, causes the entropy to approach a steady-state value that depends on the bath's characteristics. In contrast, the de- polarizing channel rapidly increases the entropy, driving the system toward a fully mixed state, where the entropy reaches its maximum value of $log\,2$. These trends highlight how different decoherence processes govern the evolution of disorder in quantum systems.

7 CONCLUSION

Our results demonstrate that the amplitude damping and thermalization processes lead to a gradual loss of energy, while the depolarizing channel causes a more abrupt energy dissipation due to rapid state mixing. Population dynamics show that amplitude damping drives the system toward the ground state, thermalization leads to equilibrium populations determined by the bath temperature, and depolarization results in a maximally mixed state. Entropy evolution reflects these trends, stabilizing for amplitude damping and thermalization but reaching its maximum in the depolarizing channel. These insights can guide the development of decoherence mitigation strategies that prioritize preventing depolarization and controlling population redistribution and entropy growth. Future work could explore the influence of environmental parameters and strategies to mitigate decoherence, enhancing quantum device robustness.

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