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# Incompatibility of General Relativity with Quantum Mechanics

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## ABSTRACT

This study is based upon identifying some of the incompatibilities of General Relativity with Quantum Mechanics and the underlying reasons for these incompatibilities. The root cause for most of the incompatibilities lies within the necessity of transition from Classical Mechanics to Quantum Mechanics. This study briefly explains the nature of this transition. The study on the motion of the vibrating non-relativistic closed string serves as the starting point of standard String Theory. According to the world view created by String Theories all known types of matter are simply vibrations of plank scaled strings. General Relativity defines a continuous smooth world view while the Quantum Mechanical world view is not so smooth. In terms of fundamentals of String Theory we can observe that this is due to the fact that the quantized string vibrates with a set of defined modes of vibrations.

## 1. INTRODUCTION

Although General Relativity is extremely accurate in explaining the universe on a large scale it fails to explain the behavior at the subatomic scale. At the subatomic scale the world view is under the rule of Quantum Mechanics. Introduction of the relativistic Quantum Mechanics by Paul Dirac in the year 1928 was significant because it can be considered as the first step trying to unify these to giant streams.

During the late 1960s and early 1970s String Theories were originally developed. They were not completely successful theories yet there were powerful features within the theories which promised hope for a theory of everything. Contributions of great physicists, mathematicians such as G. Veneziano, Youchiro Nambu, H. Nielson, J. Schwarz, M. Green and Edward Witten drastically shaped the framework of String Theory. String theories urged the existence of extra dimensions, most curled in plank scales. The understandings about Calabi-Yau spaces and Super Symmetry were significant achievements which paved the path to the formation of five independent Super String Theories. In the presence of an 11 dimensional space time framework the five theories yielded down to form a single theory which was known as the M-Theory. M-Theory described that the five string theories were not independent rather they all were describing the same theory in five different ways [1, 2].

### 1.1 The Incompatibilities of Quantum Mechanics and General Relativity

The realm of General Relativity is that of large scales and huge masses. It is a classical theory that works well in describing the behaviors of heavenly objects in a remarkably accurate manner.

According to General Relativity, the presence of mass is capable of distorting the space time fabric while the absence of mass depicts a flat space time fabric. However as the scale of the space time fabric gets smaller and smaller and reaches the subatomic level, quantum fluctuations arising from the uncertainty principle begins to dominate, even for the gravitational field. These quantum fluctuations lead to violent distortions of the space-time fabric, in sharp contrast to the smooth world view of General Relativity [2]. While analyzing these incompatibilities, it is extremely important to observe this smooth and not so smooth world views that these two streams create. Moving on to String Theory, the central feature of String theory depicts that elementary particles, even protons and neutrons are simply different vibrating modes of strings. The essence of this work is based upon understanding this aspect.

Apart from the smooth and the chunky world views that they create, there are lots of incompatibilities.

- Quantum theory is not locally causal.
- The incapability of General Relativity to handle point like masses which give positive infinity as a solution for the space-time curvature.

Incompatibilities between Quantum Mechanics and General Relativity occur mainly due to the mathematical frameworks that they are formulated in. The root cause for most of the incompatibilities lies within the necessity of transition from Classical Mechanics to Quantum Mechanics. The Quantum Mechanical commutation relation differs from the classical commutation of the same quantities by of order $i\hbar$. The non-commutative nature of matrices led to the understanding of the canonical commutation relation:

$$[\mathrm{X}, \mathrm{P}] = \mathrm{XP} - \mathrm{PX} = \mathrm{i}\hbar$$

The classical commutation of position and momentum would simply lead to;

$$[X, P_x] = XP_x - P_xX = 0$$

This serves as a very important deviation that defines the classical limits and also serves as a starting point for the incompatibilities between Classical and the Quantum pictures.

**1.2 Relativistic Wave Equations**

By replacing the E and P (Energy and Momentum) by the operators $-i\hbar\frac{\partial}{\partial t}$ and $-i\hbar\nabla$, in the relativistic energy relation, we obtain the famous:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = \frac{m^2c^2}{\hbar^2}\psi \qquad (1.1)$$

Klein Gordon equations also known as the Schrodinger’s relativistic equation. Similarly starting from the relativistic energy relation and taking the square roots on both sides, we end up with:

$$E = \pm(c^2p^2 + m^2c^4)^{\frac{1}{2}}$$

This is a step that needs to be observed with greatest importance, and it leads to:

$$i\hbar\frac{\partial\psi}{\partial t} = (c\alpha.p + \beta mc^2)\psi \qquad (1.2)$$

This is known as the Dirac equation for a free particle [3][4]. The Dirac equation serves as the starting point in the process of unifying Quantum Mechanics with Relativity. Special Relativity and Quantum Mechanic has been combined through these elegant works of Dirac and the unification of General Relativity and Quantum Mechanics is still an ongoing process.

## 2. STRINGS

The String Theory approach of describing matter at its most fundamental level is a hope to resolve the conflict between General Relativity and Quantum Mechanics. Beside the well-known incompatibility to incorporate gravity to the standard model, the standard model also fails to explain some details of its construction materials. The central feature of String Theory which states that properties of an elementary particle such as its mass, the various force charges are determined by the resonant pattern of vibration of a plank scaled string gives a striking new way of viewing nature and the underlying laws of physics. Therefore let us observe the fundamentals of String theory in the following section. Strings can be broadly classified as open strings (which has two end-points) and closed strings (which takes the form of loops). Based upon the manner in which we observe them we can classify strings as non-relativistic strings and relativistic strings [2].

### 2.1 The Wave Equation for a Non-relativistic Open String

Although the ideal study will be the analysis of relativistic strings, for simplicity we have considered the motion of non-relativistic open strings. By considering the transverse fluctuations of a stretched String, where $x$ depicts the motion along the string while $y$ depicts the motion orthogonal to the String, we end up with the usual non-relativistic open string equation or the usual wave equation. The parameter $v$ is the usual velocity of waves, where $v = \sqrt{\tau/\sigma}$ in which $\tau$ is the tension of the String and $\sigma$ is the mass per unit length [5].

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \qquad (1.3)$$

The same equation can also be derived using the Lagrangian formalism. If "$a$" was the length of the string, the Lagrangian $L$ for the string can be written in the form:

$$L = \int_0^a \left[\frac{1}{2}\sigma\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2\right] dx$$

Then by considering the Lagrangian density, the action of the String and the variation of the action"$\delta S$", we may end up in:

$$\delta S = \int_0^a \left[\sigma\frac{\partial y}{\partial t}\delta y\right]_{t_i}^{t_f} dx + \int_{t_i}^{t_f}\left[-\tau\frac{\partial y}{\partial x}\delta y\right]_0^a dt + \int_{t_i}^{t_f}\int_0^a\left(\tau\frac{\partial^2 y}{\partial x^2} - \sigma\frac{\partial^2 y}{\partial t^2}\right)\delta y dx dt$$

Since $\delta S = 0$ each of the above terms should vanish independently. In particular the third term of the above expression vanishes and thus we end up with our usual wave equation.

The second term can be made to zero in two ways; one is by setting $\frac{\partial y(t,0)}{\partial x} = \frac{\partial y(t,a)}{\partial x} = 0$, that is by specifying the values of the derivatives $\frac{\partial y}{\partial x}$ at the end points, which gives the "Neumann conditions" and another way is to make $\delta y(t,0) = \delta y(t,a) = 0$. This corresponds to "Dirichlet conditions" in which the positions of the string end points are specified [5].

### 2.2 Factorizing the Wave Equation

It is important to observe the striking resemblance between the classical wave equation and the relativistic Quantum Mechanical Klein Gordon Equation.

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

*The classical wave equation*

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = \frac{m^2 c^2}{\hbar^2}\psi$$

*Relativistic Quantum Mechanical Klein-Gordon equation*

By replacing the term "*y*" by a wave function $\psi$ and by considering the terms of the wave equation to act as operators we may obtain an equation similar to the left hand side of the Klein Gordon Equation. Keeping this in our minds let us recall the work of Dirac. As stated before the Dirac equation was obtained by taking the square root of $E^2 = c^2p^2 + m^2c^4$ which led to $E = \pm\sqrt{c^2p^2 + m^2c^4}$. by considering these we can show that the wave equation can be factorized.

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi = \left(\frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right)\psi$$

Thus the wave equation can be written as

$$\left(\frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right)\psi = 0$$

This serves as a different way of observing the wave equation.

### 2.3 The General Solution for a Non-relativistic String

The general solution under Dirichlet boundary conditions can be written as

$$y_n(t,x) = A_n \sin\left(\frac{n\pi x}{a}\right)\sin\left(\sqrt{\frac{\tau}{\sigma}}\left(\frac{n\pi}{a}\right)t + \varphi\right) \tag{1.4}$$

Where $\varphi$ is the constant common phase, $A_n$ is a constant corresponding to "$n$",which is an integer. It can be shown that the Neumann boundary conditions lead us to an equation of the form:

$$y_n(t,x) = y_0 + vt + A_n \cos\left(\frac{n\pi x}{a}\right)\sin\left(\sqrt{\frac{\tau}{\sigma}}\left(\frac{n\pi}{a}\right)t + \varphi\right) \tag{1.5}$$

For simplicity, in the following context we have only considered motions under the Dirichlet boundary condition in which the string is considered to be fixed at both ends.

### 2.4 Quantization of the Non-relativistic Strings

The wave equation:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

The solution under the Dirichlet boundary condition:

$$y_n(t,x) = A_n \sin\left(\frac{n\pi x}{a}\right)\sin\left(\sqrt{\frac{\tau}{\sigma}}\left(\frac{n\pi}{a}\right)t\right)$$

The Lagrangian density of the non-relativistic string:

$$\mathcal{L} = \frac{1}{2}\sigma\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2$$

The momentum density can be written as

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \sigma\dot{y}$$

The Hamiltonian density can be written as

$$\mathcal{H} = \Pi\dot{y} - \mathcal{L} = \frac{\Pi^2}{2\sigma} + \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2$$

By solving the Hamilton's equations of motion

$$-\dot{\Pi} = \frac{\partial \mathcal{H}}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\Pi^2}{2\sigma}\right) + \frac{1}{2}\tau\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right)^2 = 0$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial \Pi} = \frac{\Pi}{\sigma}$$

In order to quantize the string let us first write the last term of the general solution (1.4) using complex numbers

Let $\omega = \sqrt{\frac{\tau}{\sigma}\frac{\pi}{a}}$ we can write

$$\sum_{n\neq 0} A_n \sin(n\omega t) = \sum_{n>0}\left(A_n \sin(n\omega t) + A_{-n}\sin(-n\omega t)\right)$$

$$\sum_{n\neq 0} A_n \sin(n\omega t) = \sum_{n>0}\frac{1}{2i}\left(A_n e^{in\omega t} - A_n e^{-in\omega t} + A_{-n}e^{-in\omega t} - A_{-n}e^{in\omega t}\right)$$

$$= \sum_{n>0}\frac{1}{2i}\left[(A_n - A_{-n})e^{in\omega t} - (A_n - A_{-n})e^{-in\omega t}\right]$$

Let us define new complex amplitudes

$$\alpha'_n = \frac{1}{2i}(A_n - A_{-n}) \qquad \alpha'_n = (\alpha'_{-n})^*$$

Now we may write $y(t,x)$ and $\Pi(t,x)$ more generally in the form

$$y(t,x) = y_0 + \sum_{n>0} \alpha'_n \sin\left(\frac{n\pi x}{a}\right) e^{in\omega t} \tag{1.6}$$

$$\Pi(t,x) = \Pi_0 + i\sqrt{\tau\sigma} \sum_{n>0} \alpha'_n \left(\frac{n\pi}{a}\right) \sin\left(\frac{n\pi x}{a}\right) e^{in\omega t} \tag{1.7}$$

In order to Quantize let us try to write $y, \Pi, \alpha'_n$ in the operator forms and impose "same time" commutation relations.

$$[\hat{y}(t,x), \hat{\Pi}(t,x')] = \frac{i\hbar}{a}\delta(x-x')$$

$$[\hat{y}(t,x), \hat{y}(t,x')] = 0$$

$$[\hat{\Pi}(t,x), \hat{\Pi}(t,x')] = 0$$

It can be shown that the non trivial commutation relations

$$[\hat{y}_0, \hat{\Pi}_0] = \frac{i\hbar}{a}$$

$$[\hat{\alpha}_m, \hat{\alpha}_n] = -m\delta_{m,-n} \text{ holds,}$$

where

$$\hat{\alpha}_n = \frac{n\pi}{a}\hat{\alpha}'_n \quad \text{and} \quad \hat{\alpha}'_n = (\hat{\alpha}'_{-n})^\dagger \tag{1.8}$$

We may also write the Hamiltonian density

$$\mathcal{H} = \frac{\Pi^2}{2\sigma} + \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2$$

$$\mathcal{H} = \frac{\left(\Pi_0 + i\sqrt{\tau\sigma}\sum_{n>0}\alpha'_n\left(\frac{n\pi}{a}\right)\sin\left(\frac{n\pi x}{a}\right)e^{in\omega t}\right)^2}{2\sigma} + \frac{1}{2}\tau\left(\sum_{n>0}\alpha'_n\cos\left(\frac{n\pi x}{a}\right)e^{in\omega t}\left(\frac{n\pi}{a}\right)\right)^2$$

The Hamiltonian can be obtained by $H = \int \mathcal{H} dx$ and this can also be quantized. The quantization process of the Hamiltonian involves standard methods of String Theory hence we shall not perform the quantization in the present study.

### 2.5 Further Studies, Experimental Verifications and Challenges

Undoubtedly the present study only depicts the beginning of String Theory. In order to precisely study the nature of the space time fabric and the world of String Theory, one must study the motion of Relativistic Strings, Nambu-Goto actions, world sheets, D-branes, Super Symmetry and many more.

When it comes to experimental verifications, it would be an impossible task to generate the energy required to observe a single "String", yet there are many indirect means of concluding the validity of String theory. Among the many experimental verifiable evidence that String theory offers, the explanation of the three families of elementary particles in terms of the geometrical holes of Calabi-Yau spaces is simply remarkable, even though there are unresolved issues of choosing the particular space. According to Edward Witten, the pioneer of M-theory; "String Theory has the remarkable property of predicting gravity". This itself simply depicts the capabilities of this magnificent theory [2].

## 3. CONCLUSION

The study was entirely based upon understanding the fundamental incompatibilities of Quantum Mechanics and General Relativity. By constraining ourselves to a simple framework we were able to have a better overall picture of the entire problem.

Incompatibilities between Quantum Mechanics and General Relativity occur mainly due to the mathematical frameworks that they are formulated in. The root cause for most of the incompatibilities lies within the necessity of transition from Classical Mechanics to Quantum Mechanics. The Quantum Mechanical commutation relation differs from the classical commutation of the same quantities by of order $i\hbar$. This serves as a very important deviation that defines the classical limits and also serves as a starting point for the incompatibilities between Classical and the Quantum pictures.

The study on the motion of the vibrating non-relativistic closed string serves as the starting point of standard String Theory. By understanding the fact that all known matter can be explained in terms of simple vibrating strings and realizing that constrains of the variations of matter are due to the discreteness of the vibrating modes is simply the essence of this study. By quantizing the simple non relativistic string we were able to come to the conclusion that the reason for the not so smooth world view created by Quantum Mechanics and fundamentals of String Theory is due to this discreetness in the string modes. This serves as a fundamental incompatibility because the General Relativistic world view is continuous and smooth.

It is believed that further studies are necessary to strengthen the above mentioned notion. Most certainly new experimental findings and theoretical insights would provide a better understanding of the universe we live in.

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